Discrete Mathematics 311 (2011) 124–131

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On the existence of latin squares with special distribution properties

David Meier

Pilgerweg 1, 8044 Zurich, Switzerland

ARTICLE INFO

Article history:
Received 6 February 2010
Received in revised form 2 October 2010
Accepted 11 October 2010
Available online 5 November 2010

Keywords:
Latin squares
Rectangular subarrays
Conflict free access
Cayley tables of cyclic groups

ABSTRACT

An $n \times n$ latin square $(a_{r,c} | 0 \le r, c \le n-1)$ is an (s, t) latin square if every subrectangle $R_{i,j}$, $0 \le i, j \le n-1$, consisting of cells $\{a_{i+k,j+\ell} | 0 \le k < t, 0 \le \ell < s\}$, where the addition of indices is performed modulo n, contains st different elements. We show that an (s, t) latin square exists if and only if $n \ge st + t$ or $n > st$ and the product of the greatest common divisors $gcd(s, n)gcd(t, n)$ is a divisor of n.

1. Introduction

A latin square is an $n \times n$ matrix with entries from a set with n elements $\{0, 1, \ldots, n-1\}$ such that every element occurs exactly once in each row and column. We are interested in latin squares in which every subrectangle of $s \times t$ elements contains st different elements in the stronger sense of Definition 1.

Let $R_{i,j}$, $0 \le i, j \le n-1$, be the $s \times t$ rectangle consisting of the cells $(i+k, j+\ell)$, where the addition of indices is performed modulo n, with $0 \le k < t$ and $0 \le \ell < s$. The first index of cell $(i+k, j+\ell)$ denotes the row, the second the column.

Also other calculations on the indices are performed modulo n and we always assume that the result is in $0, \ldots, n-1$.

Definition 1. The latin square $(a_{r,c} | 0 \le r, c \le n-1)$ is an (s, t) square if for every rectangle $R_{i,j}$, $0 \le i, j \le n-1$ the values at the st cells are different.

We can therefore think of the latin square made into a torus or that we cover the plane with identical copies of the latin square and that the $s \times t$ rectangles can also cross borders or corners.

Because of the latin square properties, only the cases where s and t are greater than 1 are of interest, this is therefore assumed throughout the paper.

Example 1. The following 7×7 latin square is a $(2, 3)$ latin square. The two 2×3 rectangles $R_{5,2}$ and $R_{6,6}$ are also shown.

$$
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
5 & 6 & 0 & 1 & 2 & 3 & 4 \\
3 & 4 & 5 & 6 & 0 & 1 & 2 \\
1 & 2 & 3 & 4 & 5 & 6 & 0 \\
6 & 0 & 1 & 2 & 3 & 4 & 5 \\
4 & 5 & 6 & 0 & 1 & 2 & 3 \\
2 & 3 & 4 & 5 & 6 & 0 & 1
\end{array}
$$

E-mail address: davidmeier@bluewin.ch.

0012-365X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2010.10.007

If a latin square $(a_{r,c}|0 \le r, c \le n-1)$ is an (s,t) square, then obviously $n \ge st$.

Latin squares where one looks only at $s \times t$ subrectangles inside the latin square, i.e. rectangles consisting of the cells $(i+k, j+\ell)$ with $0 \le k < t$ and $0 \le \ell < s$ and with $0 \le i < n-t$ and $0 \le j < n-s$, were studied in [1,2,6] in the context of conflict free access to parallel computer memories.

The notation (s,t) square is taken from [2], even if it stands there for the weaker property above.

In [6] Van Voorhis and Morrin show that a necessary condition for the weaker property, in which one looks only at $s \times t$ subrectangles inside the latin square, is that $n > st$. This is therefore also necessary for (s,t) latin squares studied in this paper.

Colbourn and Heinrich show in [2] that the condition $n > st$ is also sufficient for the existence of a latin square for the weaker condition.

Several authors consider also other forms such as stars or diamonds, for example in [3].

The main objective of our paper is to completely determine the triples n, s, t for which there exists an (s,t) latin square of order n.

We show in Section 3 that (s,t) latin squares always exist, if $n \ge st + t$. Then we show in Section 4 that for $n = st + r$ with $0 < r < \min(s,t)$ a necessary and sufficient condition for the existence of an (s,t) latin square is that the product of the greatest common divisors $\gcd(s,n)\gcd(t,n)$ is a divisor of n.

In the final Section 5 we apply the results to Cayley tables of cyclic groups.

Our main concern in this paper is the solution of a mathematical problem. But there are applications to conflict free access to parallel computer memories. For example if a larger two-dimensional memory block is divided into subsquares which are to be accessed in parallel, then a distribution of the information in the subsquares, the same distribution for all subsquares, according to our constructions, could be useful.

2. (s, t) diagonals

Following Hedayat [4], a set D of n cells $D_k = (r_k, c_k)$, $k = 0, \ldots, n-1$ in an $n \times n$ square such that $0 \le r_k, c_k \le n-1$ and no two cells are in the same row or in the same column is called a diagonal.

Definition 2. The diagonal D is called an (s,t) diagonal if there does not exist a $s \times t$ rectangle $R_{i,j}$ containing two or more cells of the diagonal.

The cells in an (s,t) latin square with a given value m form a (s,t) diagonal.

Because in our (s,t) property we allow border and corner crossing $s \times t$ rectangles, we can shift the cells in vertical or horizontal direction in a wrap around way without losing the (s,t) property and can therefore always assume that the diagonal D begins with cell $D_0 = (0,0)$.

We can always rearrange the cells of a diagonal D, such that the ith cell is in row i, i.e. $D_i = (i, c(i))$, then an (s,t) diagonal D can be characterized in the following way.

Lemma 1. *Let D be a diagonal with $D_i = (i, c(i))$. Then D is an (s,t) diagonal if and only if for all i:*

$$s \le c(i+j) - c(i) \le n-s \quad \text{for all } j \text{ with } 0 < j < t. \quad \Box$$

To a diagonal D we can associate a latin square $L(D)$ by placing the value ℓ in cell $(i, c(i)+\ell)$ for all $0 \le \ell < n$ and cells $D_i = (i, c(i))$.

Proposition 1. *If the diagonal D has the (s,t) property, then the latin square $L(D)$ is an (s,t) latin square.*

Proof. The set D_ℓ of cells with value ℓ is obtained by shifting the diagonal D by ℓ columns, so that it inherits the (s,t) property. $\Box$

Corollary 1. *The following are equivalent. (a) There exists a latin square of size n with the (s,t) property. (b) There exists a diagonal D with the (s,t) property.*

Proof. (a) ⇒ (b). The cells D_i with value 0 form a diagonal D with the (s,t) property.

(b) ⇒ (a). Given a diagonal D with the (s,t) property, the latin square $L(D)$ has (s,t) property by Proposition 1. $\Box$

Next we construct (s,t) diagonals for two special cases.

Proposition 2. *If $n > st$ and $\gcd(n,s) = 1$, then $D_i = (i, is)$ for $0 \le i < n$ is an (s,t) diagonal.*

Proof. (a) Since $\gcd(n,s) = 1$, the set of column coordinates $\{c(i) = is, i = 0, \ldots, n-1\}$ is the set of all numbers from 0 to $n-1$, therefore D is a diagonal.

(b) $c(i+k) - c(i) = ks$, with $0 < k < t$, we get $s \le c(i+k) - c(i) \le n-s$, so that D is an (s,t) diagonal by Lemma 1. $\Box$

It is this construction, that we used to produce the $(2, 3)$ latin square of order 7 of Example 1 in Section 1.
The construction used in Proposition 2 goes back to [2] and earlier to [1].

Proposition 3. *If $n > 2s$, then there exists an $(s, 2)$ diagonal.*

Proof. (a) $n = 2\ell + 1$. Then $\ell \geq s$ and $\gcd(\ell, n) = 1$. By Proposition 2, the set D with $D_i = (i, i\ell)$ for $0 \leq i < n$ is an $(\ell, 2)$ diagonal, and therefore an $(s, 2)$ diagonal.

(b) $n = 2\ell$, ℓ even. Then $\ell - 1 \geq s$ and $\gcd(\ell - 1, n) = 1$. By Proposition 2, the set D with $D_i = (i, i(\ell - 1))$ for $0 \leq i < n$ is an $(\ell - 1, 2)$ diagonal, and therefore an $(s, 2)$ diagonal.

(c) $n = 2\ell$, ℓ odd. Then $\ell - 1 \geq s$ and $\gcd(\ell - 1, n) = 2$. Let $D_i = (i, c(i))$ and $c(i) = i(\ell - 1)$ for $0 \leq i < n/2$ and $c(i) = 1 - (i - n/2)(\ell - 1)$ for $n/2 \leq i < n$. Then the $c(i)$ are a complete set of representatives for the numbers $0, 1, 2, \ldots, n - 1$. By Lemma 1 with $s = 2$, we have to show that $\ell - 1 \leq c(i + 1) - c(i) \leq n - (\ell - 1)$. Now $c(i + 1) - c(i)$ can has one of the values $\ell - 1, \ell, n - (\ell - 1)$, such that $\ell - 1 \leq c(i + 1) - c(i) \leq n - (\ell - 1)$ holds. $\square$

3. The existence of (s, t) latin squares with $n \geq st + t$

For $n > st$ an easy way to construct a diagonal $D_i = (i, c(i)),\ i = 0, \ldots, n - 1$ is the following. Let $d = \gcd(s, n)$ and

$$\begin{array}{ll} c(i) = si & \text{for } 0 \leq i < n/d \\ c(i) = s(i - n/d) + 1 & \text{for } n/d \leq i < 2n/d \\ \ldots & \ldots \\ c(i) = s(i - (d - 1)n/d) + d - 1 & \text{for } (d - 1)n/d \leq i < n. \end{array}$$

That is, $c(i + 1)$ is obtained by adding s to $c(i)$. If this would lead to an already existing column, $c(i + 1)$ is obtained by adding $s + 1$ to $c(i)$.

We call this diagonal the coset enumeration diagonal for n and s.

The corresponding latin square has the property that $s \times t$ subrectangles inside the latin square contain st different elements and was used by Colbourn and Heinrich in [2].

For $d > 1$ the coset enumeration diagonal is however not an (s, t) diagonal in our sense, because $c(0) - c(n - 1) = s - d + 1$, which does not satisfy the condition $s \leq c(0) - c(n - 1) \leq n - s$ of Lemma 1.

For $n > st + t$ we modify the coset enumeration diagonal for n and $s + 1$ by changing the columns $c(i)$ of the diagonal above in the following way: $c(i + 1)$ is obtained by adding $s + 1$ to $c(i)$. If this would lead to an already existing column, $c(i + 1)$ is obtained by adding s to $c(i)$. In the proof of Theorem 1, it is shown, that the resulting diagonal is an (s, t) diagonal.

Theorem 1. *If $n > st + t$ then there exists an (s, t) diagonal.*

Proof. Let $d = \gcd(s + 1, n)$, then $n - t(s + 1) \geq d$.

(a) $d = 1$. Let $D_i = (i, c(i))$ with $c(i) = i(s + 1)$. Then D is an $(s + 1, t)$ diagonal by Proposition 2, and therefore also an (s, t) diagonal.

(b) $d > 1$. Let $D_i = (i, c(i))$ and let $c(i)$ be defined as follows:

For $i = k(n/d) + \ell$ with $0 \leq \ell < n/d$ and $0 \leq k < d$ we set $c(i) = n - k + \ell(s + 1)$. Thus for $i = k(n/d)$ to $i = k(n/d) + (n/d - 1)$ the $c(i)$ form the coset C_{n-k} of the multiples of $s + 1$ containing $n - k$. With $0 \leq k < d$, the set D is a diagonal.

Since $n/d > t$ there is at most one coset change in $c(i), c(i + 1), \ldots, c(i + j)$ for $0 < j < t$. Since $c(0) - c(n - 1) = n - (n - (d - 1) + (n/d - 1)(s + 1)) = s + d$ there are three possibilities for

$$c(i + j) - c(i) = \begin{cases} j(s + 1) & \text{if } c(i + j) \text{ and } c(i) \text{ are in the same coset} \\ (j - 1)(s + 1) + s & \text{if } c(i + j) \text{ and } c(i) \text{ are in the different cosets,} \\ \quad \text{but not in the pair } C_0 \text{ and } C_{n-(d-1)} & \\ (j - 1)(s + 1) + s + d & \text{if } c(i + j) \text{ is in } C_0 \text{ and } c(i) \text{ is in } C_{n-(d-1)}. \end{cases}$$

Therefore we get in all three cases $s \leq c(i+j) - c(i) \leq n - s$ for $0 < j < t$ so that D is an (s, t) diagonal by Lemma 1. $\square$

The following example illustrates the construction for $s = 5$, $t = 3$ and $n = 21$ to obtain a diagonal.

Example 2. Construction of a $(5, 3)$ diagonal $D_i = (i, c(i))$ for $n = 21$. The following table shows the $c(j + \ell)$ constructed by the process of Theorem 1.

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$
$j = 0$	0	6	12	18	3	9	15
$j = 7$	20	5	11	17	2	8	14
$j = 14$	19	4	10	16	1	7	13

The next theorem shows how to construct an (s, t) diagonal for $n = st + t$. Also in this construction we start with the coset enumeration diagonal for n and $s + 1$ and show that a rearrangement, which places the column values of the form $n - is$, $i = 1, \ldots, s$ at the end, leads to an (s, t) diagonal.

Theorem 2. *If $n = st + t$ then there exists an (s, t) diagonal.*

Proof. (a) $t > s$. Then $n > st + s$ and the result follows from Theorem 1.

(b) $t \leq s$. We construct an (s, t) diagonal:

We begin by setting $c_{i,j} = i + j(s + 1)$, $0 \leq i \leq s$, $0 \leq j < t$.

$$
\begin{array}{ccccc}
c_{0,0} & c_{0,1} & \cdots & \cdots & c_{0,t-1} \\
c_{1,0} & c_{1,1} & \cdots & \cdots & \mathbf{c_{1,t-1}} \\
\cdots & \cdots & \cdots & \mathbf{c_{2,t-2}} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{c_{t,0}} & \cdots & \cdots & \cdots & \cdots \\
c_{t+1,0} & \cdots & \cdots & \cdots & \mathbf{c_{t+1,t-1}} \\
\cdots & \cdots & \cdots & \mathbf{c_{t+2,t-2}} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
c_{s,0} & \cdots & \mathbf{c_{s,j}} & \cdots & c_{s,t-1}
\end{array}
$$

The $c_{i,j}$ with value $c_{i,j} = n - is$ for $i > 0$ are written in bold face.

Now we rearrange the $c_{i,j}$ in the following way: we omit the bold face $c_{i,j}$ and shift the remaining ones vertically up, the bold faced ones are placed in the bottom rows in reversed order in the following way, giving a partition of the numbers into three distinct subsets

$$
\begin{array}{ccccc|l}
c_{0,0} & c_{0,1} & \cdots & \cdots & c_{0,t-1} & part1 \\
\hline
c_{1,0} & c_{1,1} & \cdots & \cdots & c_{2,t-1} & part2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \\
\cline{3-6}
c_{s,0} & \cdots & \multicolumn{1}{|c}{\mathbf{c_{s,j}}} & \cdots & \cdots & part3 \\
\cline{1-2}
\cdots & \cdots & \cdots & \cdots & \cdots & \\
\mathbf{c_{2t,0}} & \cdots & \cdots & \mathbf{c_{t+2,t-2}} & \mathbf{c_{t+1,t-1}} & \\
\mathbf{c_{t,0}} & \mathbf{c_{t-1,1}} & \cdots & \mathbf{c_{2,t-2}} & \mathbf{c_{1,t-1}} &
\end{array}
$$

Now let D be the set of cells with $D_i = (i, c(i))$. If $i = i_1 t + i_2$ with $0 \leq i_2 < t$, then $c(i)$ is the term in row i_1 and column i_2 in the array above.

Because the array contains all numbers from 0 to $n - 1$, D is a diagonal.

In part 1 the difference of two consecutive terms is $s + 1$, the difference of the first term in part 2 and the last in part 1 is $s + 2$. Part 2 contains s sequences of length $t - 1$ of elements from the same coset of multiples of $s + 1$ with differences $s + 1$ and $s + 2$ if the coset changes, the difference of the first term in part 3 and the last in part 2 is $s + 1$. In part 3 the difference of two consecutive terms is s, the difference of the first term in part 1 and the last in part 3 is s. Therefore $c(i + k) - c(i)$ with $0 < k \leq t - 1$ satisfies the inequality $s \leq c(i + k) - c(i) \leq n - s$ and D is an (s, t) diagonal by Lemma 1. $\square$

The following example illustrates the process for $s = 8$, $t = 3$ and $n = 27$ to obtain a diagonal.

Example 3. Construction of a $(8, 3)$ diagonal $D_i = (i, c(i))$ for $n = 27$. The following table shows the columns constructed by the process of Theorem 2.

$$
\begin{array}{ccccccc}
0 & 9 & 18 & & c(0) = 0 & c(1) = 9 & c(2) = 18 \\
1 & 10 & \mathbf{19} & & c(3) = 1 & c(4) = 10 & c(5) = 20 \\
2 & \mathbf{11} & 20 & & c(6) = 2 & c(7) = 12 & c(8) = 21 \\
\mathbf{3} & 12 & 21 & & c(9) = 4 & c(10) = 13 & c(11) = 23 \\
4 & 13 & \mathbf{22} & \longrightarrow & c(12) = 5 & c(13) = 15 & c(14) = 24 \\
5 & \mathbf{14} & 23 & & c(15) = 7 & c(16) = 16 & c(17) = 26 \\
\mathbf{6} & 15 & 24 & & c(18) = 8 & c(19) = \mathbf{17} & c(20) = \mathbf{25} \\
7 & 16 & \mathbf{25} & & c(21) = \mathbf{6} & c(22) = \mathbf{14} & c(23) = \mathbf{22} \\
8 & \mathbf{17} & 26 & & c(24) = \mathbf{3} & c(23) = \mathbf{11} & c(26) = \mathbf{19}
\end{array}
$$

4. (s, t) latin squares for $n = st + r$, $0 < r < \min(s, t)$

We show that for $n = st + r$, $0 < r < \min(s, t)$ a necessary and sufficient condition for the existence of an (s, t) latin square is that $\gcd(s, n)\gcd(t, n)$ is a divisor of n. We derive in several steps that for $0 < r < \min(s, t)$ an (s, t) diagonal has a two-dimensional mesh structure which implies the product condition above.

First we define a number of terms that are used in this section.

Because of the addition modulo n for indices in this paper, we have to specify, what we mean, if we say that a cell C_1 is to the right of a cell C_2.

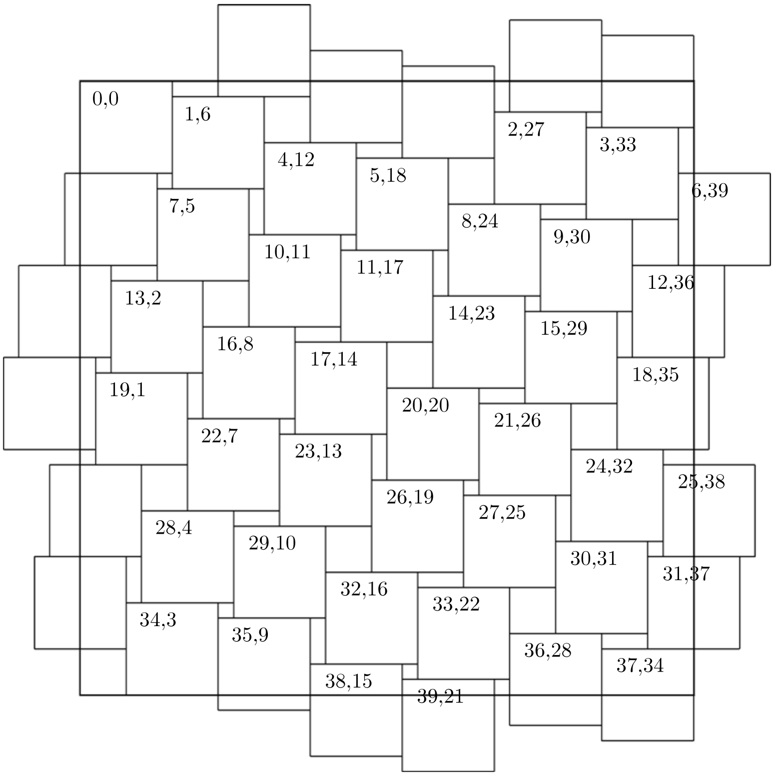

Fig. 1. A (6, 6) diagonal for $n = 40$.

Definition 3. We think of the latin square made into a torus and say that a cell $C_2 = (r_2, c_2)$ is to the right of a cell $C_1 = (r_1, c_1)$ if there are less columns to cross to reach column c_2 from column c_1 by going to the right than by going to the left. More formally:

We say that C_2 is to the right of C_1 if $c_2 - c_1 < n/2$ and that cell C_2 is to the left of C_1 if $c_2 - c_1 > n/2$.
Similarly
we say that C_2 is below C_1 if $r_2 - r_1 < n/2$ and that cell C_2 is above C_1 if $r_2 - r_1 > n/2$.

With this definition, we can define cyclic monotonicity of a sequence of cells.

Definition 4. A sequence of cells $C_0 = (r_0, c_0), \ldots, C_{k-1} = (r_{k-1}, c_{k-1})$ is called cyclically monotonous if the direction, left or right, from one cell to the next remains constant.

More formally the sequence is called right cyclically monotonous if

$$C_{i+1} \text{ is to the right of } C_i \quad \text{for } 0 \leq i < k-1 \text{ and } C_0 \text{ is to the right of } C_{k-1}$$

and it is called left cyclically monotonous if

$$C_{i+1} \text{ is to the left of } C_i \quad \text{for } 0 \leq i < k-1 \text{ and } C_0 \text{ is to the left of } C_{k-1}.$$

An example of a left cyclically monotonous sequence of cells is obtained as follows.

Example 4. Let $n = st + r$ and $0 < r < \min(s, t)$ and $\gcd(n, s) = 1$. By Proposition 2, the cells $D_i = (i, c(i))$ with $c(i) = is$ form a diagonal. The sequence of cells $C_j = D_{jt} = (jt, jst)$, $j = 0, \ldots, n/\gcd(n, t) - 1$ has the property $c_{j+1} - c_j = (j+1)st - jst = st = n - r > n/2$ and is therefore left cyclically monotonous.

Now let $D_i = (r_i, c_i)$, $i = 0, \ldots, n-1$, be a diagonal and let $R(D_i)$ be the $s \times t$ rectangle with upper left corner D_i.
Fig. 1 shows a (6, 6) diagonal $D_i = (r_i, c_i)$, $i = 0, \ldots, n-1$, for $n = 40$ with the associated 6×6 squares $R(D_i)$.
We say that two rectangles overlap if they have one or more cells in common.
Then we can characterize (s, t) diagonals in the following way.

Proposition 4. *The following are equivalent. (a) Diagonal D with $D_i = (i, c(i))$ is a (s, t) diagonal. (b) No two rectangles of the n rectangles $R(D_i)$ overlap.*

Proof. Let $D_i = (i, c(i))$ and $D_j = (j, c(j))$ be two cells of D. The condition, that D_i and D_j are contained in an (s, t) rectangle and the condition that the $s \times t$ rectangles $R(D_i)$ and $R(D_j)$ overlap, are identical: the vertical distance of the D_i and D_j is less than t and the horizontal distance is less than s, more formally $j - i < t$ or $j - i > n - t$, and, $c(j) - c(i) < s$ or $c(j) - c(i) > n - s$. □

In the next lemma it is shown, that the columns of two cells in row i and $i + t$ of an (s, t) diagonal differ by at most r.

Lemma 2. *Assume that D with $D_i = (i, c(i)), 0 \le i \le n - 1$ is a (s, t) diagonal for $n = st + r$ and $0 < r < \min(s, t)$. Then $c(i + t) - c(i) \le r$ or $c(i + t) - c(i) \ge n - r$ for $0 \le i \le n - 1$.*

Proof. Consider the $t - 1$ rectangles $R(D_{i+1}), \ldots, R(D_{i+t-1})$ of size $s \times t$. Each of them covers s cells of row $i + t$, such that there remain $s + r$ free cells in row $i + t$, of which s start at column $c(i)$. Since $r < s$, the s cells starting at $c(i)$ and the s cells starting at column $c(i + t)$ must overlap and therefore $c(i + t) - c(i) \le r$ or $c(i + t) - c(i) \ge n - r$. □

Next we define adjacency for rectangles of cells in an obvious way.

Definition 5. For $D_i = (i, c(i))$ and $D_{i+t} = (i + t, c(i + t))$ the two rectangles $R(D_i)$ and $R(D_{i+t})$ are called adjacent if $c(i + t) - c(i) < s$ or $c(i + t) - c(i) > n - s$.
Similarly
for $D_i = (i, c(i))$ and $D_{i'} = (i', c(i) + s)$ the two rectangles $R(D_i)$ and $R(D_{i'})$ are called adjacent if $i' - i < t$ or $i' - i > n - t$.

Then the following corollary is a consequence of Lemma 2 and the symmetry of the situation of for rows and columns.

Corollary 2. *If $D_i = (i, c(i)), 0 \le i \le n - 1$ is a (s, t) diagonal for $n = st + r$ and $0 < r < \min(s, t)$, then the rectangles $R(D_i)$ and $R(D_{i+t})$ are adjacent and for $D_i = (i, c(i))$ and $D_{i'} = (i', c(i) + s)$ the two rectangles $R(D_i)$ and $R(D_{i'})$ are adjacent for $0 \le i < n$.*

Every rectangle $R(D_i)$ of a diagonal D is therefore surrounded by four adjacent rectangles of the diagonal.
Let now $d_s = \gcd(n, s)$ and $d_t = \gcd(n, t)$. Then we can partition the cells of a diagonal in two ways:

Definition 6. $S_i = \{S_{i,j} = (i + jt, c(i + jt)), 0 \le j < n/d_t\}$ for $0 \le i < d_t$ and $T_i = \{T_{i,j} = (r(i + js), i + js), 0 \le j < n/d_s\}$ for $0 \le i < d_s$.

The next proposition shows that the rectangles $R(D_i)$ of an (s, t) diagonal D form a two-dimensional structure of ribbons of rectangles.

Proposition 5. *Assume D is an (s, t) diagonal for $n = st + r$ and $0 < r < \min(s, t)$ and that $D_i = (i, c(i)), D_{i+t} = (i + t, c(i + t)), D_{i'} = (i', c(i) + s)$, are three elements of an (s, t) diagonal D. Then the cell $(i' + t, c(i + t) + s)$ belongs to D, i.e. the four cells form a parallelogram.*

Proof. There exists a cell in D in column $c(i + t) + s$, say $D_j = (j, c(i + t) + s)$. According to Corollary 2 rectangle $R(D_j)$ is adjacent to rectangle $R(D_{i+t})$. Since there is no overlapping between rectangle $R(D_j)$ and rectangle $R(D_{i'})$, we get $j \ge i' + t$. Since there can be at most $r < s$ cells in row $i' + t$ not covered by the rectangles of D, we must have $j = i' + t$. □

For the next lemma, T_0 denotes the partition of diagonal D of cells $D_0 = (0, 0), D_{i_k} = (i_k, ks),\ k = 1, 2, \ldots$; see Definition 6.

Lemma 3. *If D is an (s, t) diagonal for $n = st + r$ and $0 < r < \min(s, t)$, then the following hold:*

(a) *the partitions S_i of D consisting of the cells $D(i, c(i)), D(i + t, c(i + t)), \ldots, D(i + (n/d_t - 1)t, c(i + (n/d_t - 1)t))$ are cyclically monotonous.*
(b) *Each partition S_i contains a cell of partition T_0 consisting of cells $D_0 = (0, 0), D_{i_k} = (i_k, ks),\ k = 1, 2, \ldots$.*

Proof.
(a) Consider three consecutive cells $D_i = (i, c(i)), D_{i+t} = (i + t, c(i + t))$ and $D_{i+2t} = (i + 2t, c(i + 2t))$ and assume that D_{i+t} is to the left of D_i and consider the four rectangles surrounding $R(D_{i+t})$; see Fig. 2. Then, since $R(D_{i+t})$ is to the left of rectangle $R(D_i)$, the cell of diagonal D in column $c(i + t) + s$ must be below D_{i+t}, such that D_{i+2t} in turn must be to the left of D_{i+t}.
(b) By applying the parallelogram property, Proposition 5, starting with partitions $S_0 = \{S_{0,0}, S_{0,1}, S_{0,2}, \ldots\}$ and $T_0 = \{T_{0,0}, T_{0,1}, T_{0,2}, \ldots\}$, see Definition 6, we can construct other cells of diagonal D in the following way: let $D_{i,0} = S_{0,i}$ for $i = 0, 1, 2, \ldots$ and $D_{0,j} = T_{0,j}$ for $i = 0, 1, 2, \ldots$ and for $i > 0$ and $j > 0$ let $D_{i,j}$ be the cell that makes $D_{0,0}, D_{i,0}, D_{i,j}, D_{0,j}$ into a parallelogram. If we consider the sets $D_i = \{D_{i,0}, D_{i,1}, D_{i,2}, \ldots\}$ and $D_{i+1} = \{D_{i+1,0}, D_{i+1,1}, D_{i+1,2}, \ldots\}$, then the rows of $D_{i,j}$ and $D_{i+1,j}$ differ by t and the columns by an amount less then or equal to r which depends only on i. Therefore the $s \times t$ rectangles with cells $D_{i,j}$ as upper left corner cover the latin square in such a way that no other $s \times t$ rectangle can be placed without overlapping with one of them. The construction produces therefore all cells of diagonal D and consequently each partition S_i consists of a set of cells $D_{k,j}, j = 0, 1, 2, \ldots$ for a fixed k. □

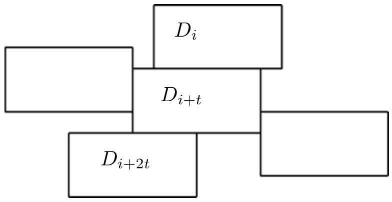

Fig. 2. Cyclic monotonicity.

We reorder the sequences in the partitions $S_i,\ i = 0, 1, \ldots$ now as follows: $S_{i,0}$ is the cell of S_i which is in T_0 i.e. $S_{i,0} = (r_i, ks)$, $S_{i,j}$ is the cell in row $r_i + jt$. Then the following holds.

Corollary 3. *Assume $S_{i,j}$ as above, then there exists a fixed $k = k(i)$ such that the columns satisfy $c_{i,j} = c_{0,j} + ks$ for some k and $0 \le j < n/d_s$.*

Proof. The column of $S_{0,0}$ is 0 and the column of $S_{i,0}$ is a multiple of s. By induction on k to Proposition 5, we can assume that any four cells $S_{0,j}, S_{0,j+1}, S_{i,j}, S_{i,j+1}$ form a parallelogram and therefore the differences of columns of $S_{0,j}, S_{0,j+1}$ and $S_{i,j}, S_{i,j+1}$ are equal. □

Theorem 3. *If $n = st + r$ with $0 < r < \min(s, t)$ and there exists an (s, t) diagonal, then $d_s d_t | n$, where $d_s = \gcd(n, s)$ and $d_t = \gcd(n, t)$.*

Proof. Let $0 \le \ell < d_s$, we consider the cells in an (s, t) diagonal D with column congruent $\ell \bmod d_s$. Since D is a diagonal, there are n/d_s of them. Because $d_s | s$ and Corollary 3 the number of such cells is the same in every partition $S_i,\ 0 \le i < d_t$ therefore the number of partitions d_t is a divisor of n/d_s. □

There exist therefore infinitely many triples s, t, n for which there is no (s, t) latin square for n, the smallest one has $n = 18$.

Example 5. There does not exist a $(4, 4)$ latin square for $n = 18$ since $d_s = \gcd(n, s) = 2$ and $d_t = \gcd(n, t) = 2$ but 4 is not a divisor of n.

More generally if s and t are even and $\min(s, t) > r = 2r'$ with r' odd, then there does not exist an (s, t) latin square for $n = st + r$.

The next theorem shows that the condition, that $\gcd(n, s)\gcd(n, t)$ is a divisor of n, is a sufficient condition for the existence of an (s, t) diagonal.

Theorem 4. *If $n = st + r, 0 < r < \min(s, t)$ and $g_s g_t | n$ where $g_s = \gcd(n, s)$ and $g_t = \gcd(n, t)$ then there exist a (s, t) diagonal of size n.*

Proof. Consider the n cells $D_{i,j,k} = (g_s g_t i + tj + k, g_s g_t i(s/g_s) + sk - j)$ with $0 \le i < n/(g_s g_t)$, $0 \le j < g_s$ and $0 \le k < g_t$.

(a) The set $D_{i,j,k}$ forms a diagonal. We look first at the row indices: $g_s g_t i + tj + k$ with $0 \le i < n/(g_s g_t)$, $0 \le j < g_s$, $0 \le k < g_t$. The row indices of the set $D_{i,j,k}$ contain all multiples of $g_s g_t$, and since $g_t = \gcd(n, t)$ and $0 \le j < g_s$ therefore also all multiples of g_t, and finally with $0 \le k < g_t$ all numbers from 0 to $n - 1$. Because $\gcd(s/g_s, n) = 1$, a similar consideration shows that the set of column indices $g_s g_t i(s/g_s) + sk - j$ contains all numbers from 0 to $n - 1$. Since we have n cells, they form a transversal.

(b) The $D_{i,j,k} = (g_s g_t i + tj + k, g_s g_t i(s/g_s) + sk - j)$ have the (s, t) property: since $n/(g_s g_t) = (s/g_s)(t/g_t) + r/(g_s g_t)$, and $\gcd(s/g_s, n/(g_s g_t)) = 1$, the cells $D_i = (i, is/g_s), 0 \le i < n/(g_s g_t)$ form an $(s/g_s, t/g_t)$ diagonal by Proposition 2. Therefore the $s/g_s \times t/g_t$ rectangles associated with the D_i do not overlap, but each is adjacent to four other rectangles. Stretching by factor $g_s g_t$ we get $n/(g_s g_t)$ cells $D'_i = (g_s g_t i, g_s g_t i(s/g_s))$ and $n/(g_s g_t)$ non-overlapping $(g_t s) \times (g_s t)$ rectangles R'_i with upper left corner D'_i in an $n \times n$ array. The rectangle adjacent to and below R'_i has upper right corner $D'_{i+t/g_t} = (g_s g_t(i + t/g_t), g_s g_t(i + t/g_t)(s/g_s))$ and is r columns to the left of the rectangle R'_i. The rectangle adjacent to and right of R'_i has upper right corner $D'_{i+1} = (g_s g_t(i + 1), g_s g_t(i + 1)(s/g_s))$ and is $g_s g_t$ and therefore at most r rows below the rectangle R'_i. Since $r < \min(s, t)$ the following construction works without overlapping: we replace each $(g_t s) \times (g_s t)$ rectangle with $g_s g_t$ rectangles of size $s \times t$ by selecting the $D_{i,j,k}$ as upper left corners. Since $r < \min(s, t)$ the new rectangles do not overlap. □

Fig. 3 illustrates the construction of Theorem 4 with $n = 14 \times 15 + 12$, $s = 14$, $t = 15$ and therefore $g_s = 2$ and $g_t = 3$. On the left it shows two 7×5 rectangles stretched by a factor of 6, i.e. 42×30 rectangles. The replacement of each of them by six 14×15 rectangles is shown on the right.

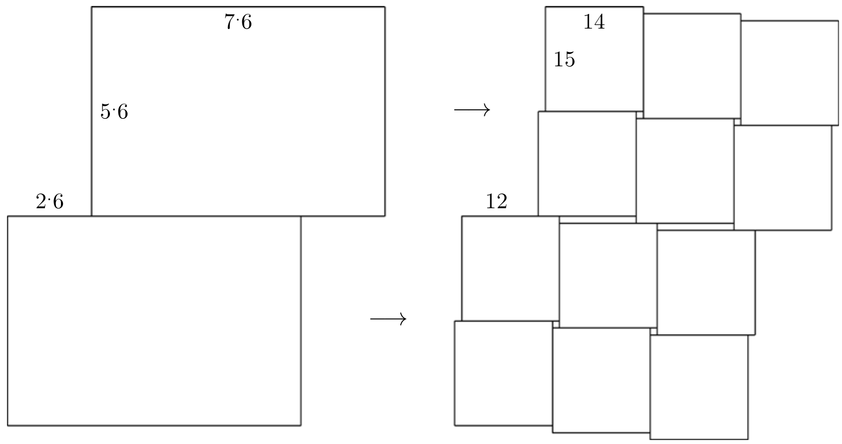

Fig. 3. Construction of a (14, 15) diagonal for $n = 14 \times 15 + 12$.

5. Cayley tables of cyclic groups

If G is a group, we say that G has the (s, t) property, if there exist enumerations of its elements $h = (g_0, g_1, \dots, g_{n-1})$ and $v = (g_{\pi(0)}, g_{\pi(1)}, \dots, g_{\pi(n-1)})$, where π is a permutation of $0, 1, \dots, n-1$, such that the Cayley table with h as horizontal border and v as vertical border and entries $g_{i,j} = g_{\pi(i)}g_j$

$$\begin{array}{r|cccc} & g_0 & g_1 & \cdots & g_{n-1} \\ \hline g_{\pi(0)} & g_{0,0} & g_{0,1} & \cdots & g_{0,n-1} \\ g_{\pi(1)} & g_{1,0} & g_{1,1} & \cdots & g_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{\pi(n-1)} & g_{n-1,0} & g_{n-1,1} & \cdots & g_{n-1,n-1} \end{array}$$

is an (s, t) latin square. In [5] we studied groups with the (2, 2) property.

The construction in Proposition 1 to construct a latin square from a diagonal actually produces a Cayley table of a cyclic group of order n, with $h = (0, 1, 2, \dots, n-1)$ and where v consists of the first column of the latin square.

We therefore get the following corollary:

Corollary 4. *A cyclic group C_n of order n has the (s, t) property with s and t greater than 1 if and only if (a) $n \geq st + t$ or (b) $n > st$ and the product $\gcd(s, n)\gcd(t, n)$ is a divisor of n.*

Acknowledgements

I would like to thank the referees for their constructive comments to the paper which have improved its presentation.

References

[1] P. Budnik, D.J. Kuck, The organization and use of parallel memories, IEEE Trans. Comput. C-20 (1971) 1566–1569.
[2] C.J. Colbourn, K. Heinrich, Conflict free access to parallel memories, J. Parallel Distrib. Comput. 14 (1992) 193–200.
[3] D.L. Galli Erickson, C.J. Colbourn, Conflict free access to constant-perimeter rectangular subarrays, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 21 (1995) 105–124.
[4] A. Hedayat, A complete solution to the existence and nonexistence of Knut Vik designs and orthogonal Knut Vik designs, J. Combin. Theory Ser. A 22 (1977) 331–337.
[5] D. Meier, On groups with the Lohse property, European J. Combin. 27 (2006) 1042–1053.
[6] D.C. Van Voorhis, T.H. Morrin, Memory systems for image processing, IEEE Trans. Comput. C-27 (1978) 113–125.